

Topological black holes in pure Gauss-Bonnet gravity and phase transitions

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March 18, 2016

Abstract

We study charged, static, topological black holes in pure Gauss-Bonnet gravity in asymptotically AdS space. As in general relativity, the theory possesses a unique nondegenerate AdS vacuum. It also admits charged black hole solutions which asymptotically behave as the Reissner-Nordström AdS black hole. We discuss black hole thermodynamics of these black holes. Then we study phase transitions in a dual quantum field theory in four dimensions, with the Stückelberg scalar field as an order parameter. We find in the probe limit that the black hole can develop hair below some critical temperature, which suggests a phase transition. Depending on the scalar coupling constants, the phase transition can be first or second order. Analysis of the free energy reveals that, comparing the two solutions, the hairy state is energetically favorable, thus a phase transition will occur in a dual field theory.

1 Introduction

It has been generally accepted that a higher-derivative gravity provides corrections to general relativity, especially in the framework of AdS/CFT correspondence [1, 2, 3], where

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large N corrections are of this form in a dual quantum field theory. If one requires that the equations of motion contain at most second order derivatives, its most general form in higher dimensions is given by the Lovelock-Lanczos action [4, 5]. It is a sum of dimensionally continued Euler densities, each one of order $p \geq 0$ in the curvature, and is characterized by a set of the coupling constants $\{\alpha_p\}$. This generalization of gravity has some distinguished properties. The higher-order terms contribute to the dynamics only in $D \geq 2p+1$ dimensions and they are free of ghosts in the flat background [6]. They also possess black hole solutions whose thermodynamic properties have been extensively studied in the literature, for example, in the Einstein-Gauss-Bonnet case [7] and, more generally, Chern-Simons gravity [8] and different Lovelock theories [9, 10, 11]. The coupling constants can be fixed requiring a unique vacuum in the theory [12]. One way to have a sensible theory is to ensure a single vacuum in any dimension. In that way, one obtains the Einstein-Hilbert action, with only the linear term in the curvature nonvanishing (the coupling α_1) and the cosmological constant (α_0). When higher-order terms are present, it gives rise to Chern-Simons gravity [13] in odd dimensions and Born-Infeld gravity [8] in even dimensions; but these vacua are $[(D-1)/2]$ -fold degenerate, which does not allow a perturbative analysis around it.

Another way to have higher-order curvature terms and the unique vacuum in Lovelock gravity which is not degenerate, is to keep only one Euler term of order p in Riemann curvature and the cosmological constant; then one obtains pure Lovelock gravity [14, 15]. This gravity possesses black hole solutions which asymptotically coincide with the corresponding Einstein solution in spite of the more complicated field equations [14, 15]. It has also been noted that a particular class of pure Lovelock black holes with the maximal order of curvature, $p = [(D-1)/2]$, has thermodynamical parameters with the universal behavior in terms of the event horizon radius [16]. Interestingly, black strings and branes cannot be constructed in general Lovelock theories, but they can in a pure Lovelock case [17, 18].

This intriguing behavior of pure Lovelock black holes motivates us to use the AdS/CFT prescription to study a holographic quantum field theory dual to pure Lovelock gravity. AdS/CFT duality links an asymptotically anti-de Sitter(AdS) gravitational bulk theory to a dual quantum theory living on its boundary. Recent progress on this subject indicates that the duality plays an important role in studying various strongly coupled phenomena in condensed matter physics [19], especially in building a gravitational dual model for a superconductor with either an electric [20] or a magnetic field [21, 22, 23, 24] (see review papers [25, 26]).

Our motivation allows us to benefit in two aspects. On one hand, the presence of higher-order terms in gravity implies, in the context of AdS/CFT correspondence, an appearance of new couplings among quantum operators in a holographic conformal field

theory. Thus, higher-curvature interaction in pure Lovelock gravity is also expected to show new features in a dual field theory. It has already been observed that there are holographic s-wave [27] and p-wave [28, 29] phase transitions in a superconductor dual to higher-order gravity, such as Einstein-Gauss-Bonnet superconductor [30, 31, 32], and the field theories dual to quasitopological gravity [33, 34], as well as Lovelock gravity [35, 36]. Numerous work on this topic confirms that higher-order terms indeed have a notable effect on phase transitions, as they modify previously universal behavior of holographic theories. On the other hand, the study of phase transitions of a superconductor via AdS/CFT duality can show some insight into the (in)stabilities of black holes. In particular, the stability of black holes in pure Lovelock gravity has been discussed in [37].

In this paper, we first add an electric charge to the pure Lovelock solutions [14, 15] and we obtain charged black holes whose asymptotic behavior is similar to the one of the Reissner-Nordström (A)dS black holes. In the AdS case, only topological black holes with hyperbolic horizons will form. Then we focus on the simplest case of pure Gauss-Bonnet AdS gravity in five dimensions and couple it to the electromagnetic and massive Stückelberg scalar fields. We explore the thermodynamics of the black holes and the possible phase transitions affected by the Stückelberg correction with backreaction. The physical explanation of condensation in this kind of superconductors is the breaking of the Abelian-Higgs mechanism. A gapless superconductor with hyperbolic geometry has been studied in Ref. [38], in which the mechanism of condensation is due to the coupling.

The rest of the paper is organized as follows. In Section 2, we review the pure Lovelock gravity and obtain a charged black hole solution. Then we introduce a holographic setup of a superconductor by coupling charged pure Gauss-Bonnet AdS gravity with a Stückelberg complex scalar in Section 3. Next, the black hole thermodynamics including the Gibbs free energy, conserved charges and quantum statistical relation are analytically investigated. Section 5 shows our numerical results of two phases and the free energy in the case of a probe limit. We close with the conclusions and discussion.

2 pure Lovelock gravity

pure Lovelock (PL) gravity action in $d + 1$ dimensions possesses, apart from the cosmological constant, a kinetic term which is a single Lovelock term of order p in the Riemann curvature [15],

$$I_{\text{PL}} = -\frac{1}{2\kappa} \int d^{d+1}x \sqrt{-g} \left(\frac{1}{2^p} \delta_{\nu_1 \dots \nu_{2p}}^{\mu_1 \dots \mu_{2p}} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \dots R_{\mu_{2p-1} \mu_{2p}}^{\nu_{2p-1} \nu_{2p}} - 2\Lambda \right). \quad (2.1)$$

A polynomial in the curvature is a $2p$ -dimensional Euler invariant continued to $d+1$ dimensions. The tensor $\delta_{\nu_1 \dots \nu_{2p}}^{\mu_1 \dots \mu_{2p}} = \det [\delta_{\nu_1}^{\mu_1} \dots \delta_{\nu_{2p}}^{\mu_{2p}}]$ is the antisymmetric Kronecker delta of rank

$2p$ and Λ is the cosmological constant whose units are $(\text{length})^{-2p}$, and the gravitational constant κ has units $(\text{length})^{d+1-2p}$. In our notation, the metric field $g_{\mu\nu}(x)$ is mostly positive and the Riemann curvature reads $R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\lambda\alpha} \Gamma^\lambda_{\nu\beta} - \Gamma^\mu_{\lambda\beta} \Gamma^\lambda_{\nu\alpha}$.

When $p = 1$, the action describes Einstein-Hilbert gravity with the (usual) cosmological constant of dimension $(\text{length})^{-2}$. When $2 \leq p \leq [d/2]$, the theory becomes pure Lovelock gravity. Thus, the simplest theory of this type is pure Gauss-Bonnet gravity in five dimensions which contains the Gauss-Bonnet term $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$.

Equations of motion in PL gravity read

$${}^{(p)}G^\mu_\nu \equiv -\frac{1}{2^{p+1}} \delta^{\mu\mu_1\cdots\mu_{2p}}_{\nu\nu_1\cdots\nu_{2p}} R^{\nu_1\nu_2}_{\mu_1\mu_2} \cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} + \delta^\mu_\nu \Lambda = 0, \quad (2.2)$$

where ${}^{(p)}G_{\mu\nu}$ is a generalized Einstein tensor. As in any Lovelock gravity, they are second order field equations in the metric.

A particular solution of these equations is the maximally symmetric spacetime with constant scalar curvature. This is flat space when $\Lambda = 0$ and $R^{\mu\nu}_{\alpha\beta} = \pm \frac{1}{\ell^2} \delta^{\mu\nu}_{\alpha\beta}$ when $\Lambda \neq 0$, corresponding to dS, (sign +) or AdS space (sign -). The effective (A)dS radius ℓ is related to a nonvanishing cosmological constant as

$$\Lambda = \frac{(\pm 1)^p d!}{2(d-2p)! \ell^{2p}}. \quad (2.3)$$

Note that (A)dS space is not directly related to the sign of the cosmological constant, as happens in general relativity, because the definition of (A)dS space is associated with the sign of curvature, and not of Λ . In five dimensions, for example, pure Gauss-Bonnet gravity ($p = 2$) has a positive cosmological constant, $\Lambda = 12/\ell^4$, and the curvature of the maximally symmetric vacuum can be either positive or negative. Indeed, writing the generalized Einstein tensor (2.2) in the form

$${}^{(2)}G^\mu_\nu = -\frac{1}{8} \delta^{\mu\mu_1\cdots\mu_4}_{\nu\nu_1\cdots\nu_4} \left(R^{\nu_1\nu_2}_{\mu_1\mu_2} - \frac{1}{\ell^2} \delta^{\nu_1\nu_2}_{\mu_1\mu_2} \right) \left(R^{\nu_3\nu_4}_{\mu_3\mu_4} + \frac{1}{\ell^2} \delta^{\nu_3\nu_4}_{\mu_3\mu_4} \right), \quad (2.4)$$

it is clear that the vacuum ${}^{(2)}G^\mu_\nu = 0$ can be in either dS or AdS space. In general, for even p , the generalized Einstein tensor always has the form ${}^{(p)}G = (R - \frac{1}{\ell^2} \delta^2) (R + \frac{1}{\ell^2} \delta^2) \mathcal{P}(R)$, where the polynomial \mathcal{P} in the curvature does not have real roots. When p is odd, then ${}^{(p)}G = (R \pm \frac{1}{\ell^2} \delta^2) \mathcal{P}(R)$ has exactly one real root. Thus, there is always at most one (A)dS vacuum, and it is not degenerate, which is suitable to study a class of asymptotically (A)dS spacetimes.

We show next that these spacetimes possess static charged black hole solutions. We focus on the case with $\Lambda \neq 0$.

2.1 Exact charged black hole solutions

Consider the pure Lovelock action coupled to a Maxwell field $A_\mu(x)$ whose field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$I_{\text{PL}} + I_{\text{M}} = -\frac{1}{2\kappa} \int d^{d+1}x \sqrt{-g} \left(\frac{1}{2^p} \delta_{\nu_1 \dots \nu_{2p}}^{\mu_1 \dots \mu_{2p}} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \dots R_{\mu_{2p-1} \mu_{2p}}^{\nu_{2p-1} \nu_{2p}} - 2\Lambda - \frac{1}{4e^2} F^2 \right). \quad (2.5)$$

A matter source in the gravitational equations of motion is the electromagnetic energy-momentum tensor,

$$T_{\mu\nu} = \frac{1}{2e^2} \left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F^2 \right), \quad (2.6)$$

and the field equations read

$$\begin{aligned} {}^{(p)}G_{\mu\nu} &= T_{\mu\nu}, \\ \nabla_\mu F^{\mu\nu} &= 0. \end{aligned} \quad (2.7)$$

We take a static and spherically symmetric ansatz for the Maxwell field, $A_\mu = \delta_\mu^t \phi(r)$, as well as for the metric $g_{\mu\nu}$,

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad (2.8)$$

where the transversal section $d\Omega_{d-1}^2$ is the maximally symmetric space of the unit radius whose curvature is $k = 1$ for dS space or $k = 0, \pm 1$ for AdS space.

It is worth noticing that the most general spherically symmetric ansatz possesses two independent metric functions $g_{tt} = -f(r)$ and $g_{rr} = 1/N(r)f(r)$ instead of (2.8) and that, in higher-order gravities, it cannot be taken for granted that field equations would uniquely solve them. In Lovelock gravity this happens only when the couplings are such that the theory possesses a degenerate vacuum [39], as, for example, in Chern-Simons (super-)gravity [40]. In our case, however, it is clear from Eq.(2.4) and the discussion below it, that the vacuum is always nondegenerate, so $g_{tt}(r)$ and $g_{rr}(r)$ are dynamical functions. This will be explicitly shown in Sec. 3.1.

In the ansatz (2.8), the equations of motion become

$$\begin{aligned} 0 &= \frac{(-1)^{p-1}(d-1)!}{2(d-2p)!r^{2p}} (f-k)^{p-1} [prf' + (d-2p)(f-k)] + \Lambda + \frac{\phi'^2}{4e^2}, \\ 0 &= \phi'' + \frac{d-1}{r} \phi', \end{aligned} \quad (2.9)$$

where the prime stands for d/dr . A solution for the electric potential is

$$\phi(r) = \mu - \frac{\rho}{r^{d-2}}, \quad (2.10)$$

where $\mu = \phi(\infty)$ is the chemical potential and ρ is an integration constant related to the electric charge. The metric function satisfies

$$(f - k)^p = \left(\mp \frac{r^2}{\ell^2} \right)^p - \frac{M_0}{r^{d-2p}} - \frac{\rho^2}{c e^2 r^{2d-2p-2}}, \quad (2.11)$$

where M_0 is an integration constant related to the mass and $c = \frac{2(-1)^p(d-1)(d-3)!}{(d-2p)!}$ is a number.

Taking the p -th root of the above equation leads to

$$f(r) = k \mp \frac{r^2}{\ell^2} \left[1 - (\mp 1)^p \left(\frac{M_0 \ell^{2p}}{r^d} + \frac{\rho^2 \ell^{2p}}{c e^2 r^{2d-2}} \right) \right]^{\frac{1}{p}}. \quad (2.12)$$

The scalar curvature behaves as $R \sim (M_0/r^d)^{1/p}$, so the spacetime of the form (2.12) has a singularity in the origin, $r = 0$, when $M_0 \neq 0$ and $p > 0$. To avoid a naked singularity, we require that it is protected by the black hole horizon, r_+ , which is the largest root of the equation $f(r_+) = 0$, and also that $f(r) \geq 0$ outside the black hole ($r \geq r_+$).

In dS space, the transversal section is always spherical, $k = 1$, and the black hole generally exists. In AdS space, only black holes with flat ($k = 0$) and hyperbolic ($k = -1$) horizons are formed.

In the asymptotic region ($r \rightarrow \infty$), without electric charge ($\rho = 0$), a black hole behaves in the leading order as the Schwarzschild-(A)dS with the mass $M_S = (\mp 1)^{p-1} M_0 \ell^{2p-2}/p$. Turning on the electric charge, a black hole with $\Lambda \neq 0$ is an asymptotically Reissner-Nordström (RN) (A)dS with the charge $Q_{RN}^2 = \rho^2 \ell^{2p-2}/e^2 p$,

$$f \sim k \mp \frac{r^2}{\ell^2} - \frac{M_S}{r^{d-2}} + \frac{Q_{RN}^2}{c r^{2d-4}}. \quad (2.13)$$

This solution is a generalization of the black holes discussed in Refs. [14, 15] to the electrically charged ones with nonspherical horizons. Since the sign of mass M_S can be positive or negative, and also the electrostatic energy can decrease the total energy of the black hole (when c is negative), it may not be stable. Thermodynamic stability of neutral pure Lovelock black holes has been discussed in Ref. [14] and recently in Ref. [37].

In what follows, we explore thermal field theories dual to PL black holes in the framework of gauge/gravity duality. We point out that, even though asymptotic behaviors of Schwarzschild and PL black holes are similar, the dynamics of respective spacetimes are different and their holographic theories will be different, as well.

2.1.1 Black holes in asymptotically AdS space

For studying thermal field theories dual to PL gravity, we are interested in AdS black holes (2.12) with noncompact horizons $k = 0$ and $k = -1$,

$$f(r) = k + \frac{r^2}{\ell^2} \left(1 - \frac{M_0 \ell^{2p}}{r^d} - \frac{\rho^2 \ell^{2p}}{c e^2 r^{2d-2}} \right)^{\frac{1}{p}}. \quad (2.14)$$

Let us first consider the planar case with $k = 0$. Without electric charge, the horizon $r_+ = (M_0 \ell^{2p})^{\frac{1}{d}}$ is formed when $M_0 > 0$. However, in that case the Hawking temperature becomes infinite for PL black holes ($p > 1$),

$$T = \frac{f'(r_+)}{4\pi} \sim \frac{1}{f(r_+)^{p-1}} \rightarrow \infty. \quad (2.15)$$

Infinite temperature is due to the fact that the scalar curvature is singular on the horizon, because we have $R \propto f(r_+)^{1-2p}$. In that case the temperature formula (2.15) might not be applicable, as it considers only the singularity at $r = 0$. Since the spacetimes with singularity horizons are not described within the standard framework of AdS/CFT correspondence in asymptotically AdS spaces, we are not interested in these cases.

Another possibility for having a noncompact horizon is to look at a hyperbolic geometry whose horizon curvature is $k = -1$. Then a neutral black hole has a horizon that satisfies the equation $f(r_+) = 0$, or

$$r_+^d - r_+^{d-2p} - M_0 \ell^{2p} = 0. \quad (2.16)$$

An existence of r_+ depends on dimension, M_0 and p . When it exists, the temperature is finite and it behaves as

$$T \sim r_+^{2p-1}, \quad (2.17)$$

which is suitable for holographic studies and high temperatures correspond to large black holes.

Adding the electric charge to the black hole is equivalent to the shift $M_0 \rightarrow M_0 + \rho^2 / c e^2 r^{d-2}$. The horizon again exists when $k = -1$ and it satisfies the equation

$$\frac{r_+^{2d-2}}{\ell^{2p}} - r_+^{2d-2p-2} - M_0 r_+^{d-2} - \frac{\rho^2}{c e^2} = 0. \quad (2.18)$$

Since the only condition is $d \geq 2p$, solutions of the above polynomial depend on particular values of the coefficients and degree of the polynomial and should be solved case by case.

On the other hand, electric properties of the charged black holes are described by the electric potential. Regularity of the potential on the horizon requires $\phi(r_+) = 0$, which relates the electric charge with the chemical potential as $\rho = \mu r_+^{d-2}$, leading to

$$\phi(r) = \frac{\rho}{r_+^{d-2}} \left(1 - \frac{r_+^{d-2}}{r^{d-2}} \right). \quad (2.19)$$

2.1.2 pure Gauss-Bonnet AdS black hole

Consider the simplest case of pure Gauss-Bonnet (PGB) gravity in five dimensions, with the Gauss-Bonnet term as the kinetic term and the cosmological constant $\Lambda = 12/\ell^4$. The PGB action with $p = 2$ in Eq. (2.1) is

$$I_G = -\frac{1}{2\kappa} \int d^5x \sqrt{-g} \left(R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4R^{\mu\nu} R_{\mu\nu} + R^2 - 2\Lambda \right), \quad (2.20)$$

and we couple it to the Maxwell field. An AdS solution for the metric function is

$$f(r) = -1 + \frac{r^2}{\ell^2} \sqrt{1 - \frac{M_0 \ell^4}{r^4} - \frac{\rho^2 \ell^4}{6e^2 r^6}}, \quad (2.21)$$

which is just Eq. (2.14) with $p = 2$ and $k = -1$. This is the only way to have a PGB AdS black hole, as discussed in Sec. 2.1.1. The black hole horizon equation (2.18) in this case can be reduced to a polynomial of third order in the positive variable $x = \left(\frac{\rho^2 \ell^4}{6e^2}\right)^{-1/3} r_+^2$,

$$x^3 - a x - 1 = 0, \quad a = (M_0 + 1) \ell^4 \left(\frac{\rho^2 \ell^4}{6e^2}\right)^{-2/3}, \quad (2.22)$$

where the real coefficient a depends on the black hole parameters. The polynomial can have three real roots in general. Analyzing the roots as a function of the parameter a , we find that the horizon forms for all values of M_0 . Depending on the value of a , the polynomial can have three zeros ($a > 1, 89$), two zeroes ($a = 1, 89$) or one zero ($a < 0$), but only one of them has a positive x corresponding to a real horizon r_+ , as shown in Figure 1. This implies that the black hole has only one horizon for any (M_0, ρ) . We conclude that there are no naked singularities. A black hole always forms, requiring $M_0 \neq 0$ for the neutral objects. The zero mass black hole exists only if $\rho \neq 0$.

Using the standard method, we calculate the Hawking temperature of the obtained black holes,

$$T = \frac{1}{4\pi} \left(\frac{2r_+^3}{\ell^4} + \frac{\rho^2}{6e^2} \frac{1}{r_+^3} \right). \quad (2.23)$$

The temperature is finite and it behaves as $T \sim r_+^3$ for large horizon radii. Note that the temperature of the PGB AdS black hole is very different from that of the RN black hole, $T_{RN} \sim r_+$, which is one of consequences of working with a higher-curvature theory. Another one is that the entropy is not proportional to the horizon area, but to the horizon line element.

The black hole temperature (2.23) is never zero. Thus, there are no extremal PGB AdS black holes.

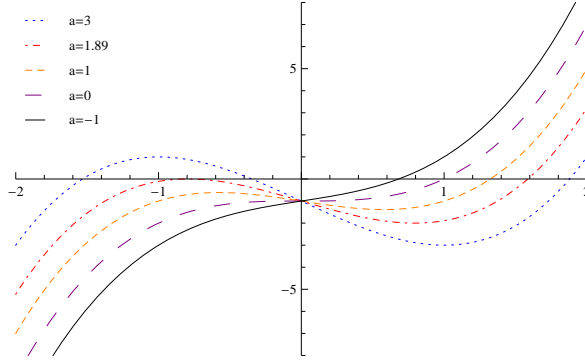


Figure 1: Behaviour of the third order polynomial $x^3 - ax - 1 = 0$ in terms of the variable $x = \left(\frac{\rho^2 \ell^4}{6e^2}\right)^{-1/3} r_+^2$ for different values of the parameter $a = (M_0 + 1) \ell^4 \left(\frac{\rho^2 \ell^4}{6e^2}\right)^{-2/3}$. There is always only one positive zero x_+ that corresponds to a unique (real) black hole horizon r_+ .

3 Pure Gauss-Bonnet gravity coupled to scalar field

In this section we explore solutions of pure Gauss-Bonnet gravity coupled to charged matter. We will focus on five-dimensional ($d = 4$) bulk gravity for simplicity, where $\Lambda = 12/\ell^4$ follows from Eq. (2.3) and the Gauss-Bonnet term ($p = 2$) is the kinetic term for gravity. We are interested in holographic applications, so we focus on asymptotically AdS space and a gravitational field coupled to a so-called Stückelberg holographic superconductor [41], built from the Maxwell electromagnetic field, A_μ , and the complex Stückelberg scalar, $\hat{\Psi} = \Psi e^{i\theta}$, where $\Psi(x)$ and $\theta(x)$ are real fields. Then the matter action is described by

$$I_M = -\frac{1}{2\kappa} \int d^5x \sqrt{-g} \left[-\frac{1}{4e^2} F^2 - \frac{1}{2} (\partial\Psi)^2 - \frac{m^2}{2} \Psi^2 - \frac{1}{2} \mathcal{F}(\Psi) (\partial\theta - A)^2 \right], \quad (3.1)$$

where a gauge invariant function $\mathcal{F}(\Psi) = \Psi^2 + c_3 \Psi^3 + c_4 \Psi^4$ is positive to ensure positivity of the kinetic term for the field θ . The minimal coupling corresponds to $c_3 = c_4 = 0$. The total action of the system is

$$I = I_0 + B = I_G + I_M + B, \quad (3.2)$$

where B is a boundary term to be added to the bulk term $I_0 = I_G + I_M$, so that the action principle for given boundary conditions is satisfied.

3.1 Equations of motion

Variation of the total action leads to the gravitational equations of motion,

$${}^{(2)}G^\mu_\nu = T^\mu_\nu. \quad (3.3)$$

The generalized Einstein tensor $^{(2)}G_\nu^\mu = H_\nu^\mu + \Lambda \delta_\nu^\mu$ includes, apart from the cosmological constant, the Lanczos tensor

$$\begin{aligned} H_\nu^\mu &= -\frac{1}{8} \delta_{\nu\nu_1\cdots\nu_4}^{\mu\mu_1\cdots\mu_4} R_{\mu_1\mu_2}^{\nu_1\nu_2} R_{\mu_3\mu_4}^{\nu_3\nu_4}, \\ &= -\frac{1}{2} \delta_\nu^\mu (R^2 - 4R^{\alpha\beta} R_{\alpha\beta} + R^{\alpha\beta\lambda\sigma} R_{\alpha\beta\lambda\sigma}) \\ &\quad + 2(RR_\nu^\mu - 2R^{\mu\lambda} R_{\lambda\nu} - 2R_{\lambda\nu\sigma}^\mu R^{\lambda\sigma} + R^{\mu\alpha\lambda\sigma} R_{\nu\alpha\lambda\sigma}). \end{aligned} \quad (3.4)$$

The energy-momentum tensor reads

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2e^2} \left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F^2 \right) + \frac{1}{2} \partial_\mu \Psi \partial_\nu \Psi + \frac{1}{2} \mathcal{F}(\Psi) (\partial_\mu \theta - A_\mu) (\partial_\nu \theta - A_\nu) \\ &\quad - \frac{1}{4} g_{\mu\nu} [(\partial\Psi)^2 + m^2 \Psi^2 + \mathcal{F}(\Psi) (\partial\theta - A)^2]. \end{aligned} \quad (3.5)$$

The Maxwell and Klein-Gordon equations are, respectively,

$$\begin{aligned} \nabla_\nu F^{\nu\mu} &= -e^2 \mathcal{F}(\Psi) (\nabla^\mu \theta - A^\mu), \\ (\square - m^2) \Psi &= \frac{1}{2} \mathcal{F}'(\Psi) (\nabla\theta - A)^2. \end{aligned} \quad (3.6)$$

The field equation with respect to $\theta(x)$ is not independent due to a local $U(1)$ symmetry. We will choose the gauge fixing $\theta = 0$.

Consider a static, topological black hole metric in AdS space that generalizes (2.8), keeping g_{rr} and g_{tt} as independent functions due to the presence of scalar fields,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)N(r)} + r^2 d\Omega^2. \quad (3.7)$$

Here, $d\Omega^2 = \gamma_{mn}(y) dy^m dy^n$ is a metric of the transversal section of the unit radius and constant curvature k . The boundary is placed at radial infinity, $r \rightarrow \infty$.

As we pointed out in Sec. 2.1.1, we will choose a hyperbolic horizon, $k = -1$. One possible choice of the transversal coordinates is

$$d\Omega^2 = d\eta^2 + \sinh^2 \eta (d\theta^2 + \sin^2 \theta d\varphi^2), \quad \eta \geq 0, \theta \in [0, \pi], \phi \in [0, 2\pi]. \quad (3.8)$$

The horizon $r_+ > 0$ is the largest root of the equation $f(r_+) = 0$ such that $N(r_+) \neq 0$ and f, N are positive functions when $r > r_+$. The black hole temperature is calculated from

$$T = \frac{1}{4\pi} f'(r_+) \sqrt{N(r_+)}. \quad (3.9)$$

As usual, we assume that all fields possess the same isometries, that is, they are static and spherically symmetric. Thus, the scalar field has the form $\Psi = \Psi(r)$, and the gauge

field, $A_\mu = \phi(r) \delta_\mu^t$, generates electric field $F_{tr} = -\phi'$ and $F^2 = -2N\phi'^2$. With this ansatz in hands, we can write the nonvanishing components of Eqs. (3.3) and (3.6) as

$$0 = -\frac{6fN'(fN-k)}{r^3} + \frac{1}{2}fN\Psi'^2 + \frac{\mathcal{F}\phi^2}{2f}, \quad (3.10)$$

$$0 = -\frac{6Nf'(fN-k)}{r^3} + \Lambda + \frac{N\phi'^2}{4e^2} - \frac{\mathcal{F}\phi^2}{4f} + \frac{1}{4}m^2\Psi^2 - \frac{1}{4}fN\Psi'^2, \quad (3.11)$$

$$0 = \frac{3Nf\Psi'}{r} + \frac{N'f\Psi'}{2} + Nf'\Psi' + Nf\Psi'' - m^2\Psi + \frac{\phi^2\mathcal{F}'}{2f}, \quad (3.12)$$

$$0 = \frac{3N\phi'}{r} + \frac{\phi'N'}{2} + N\phi'' - \frac{e^2\mathcal{F}\phi}{f}. \quad (3.13)$$

Independent gravitational equations are along rr and tt . Note that Eq. (3.10) is a difference of the original equations that arises from a backreaction of the gravitational field, leading to $g_{rr}g_{tt} \neq -1$ ($N \neq 1$) and in general different rr and tt energy-momentum components,

$$T_t^t - T_r^r = -\frac{1}{2}fN\Psi'^2 - \frac{\mathcal{F}\phi^2}{2f}. \quad (3.14)$$

Without matter fields, $T_t^t = T_r^r$ and Eq.(3.10) uniquely gives $N(r) = 1$. When the matter is present, the energy-momentum tensor has to satisfy the weak energy condition $T_{\mu\nu}u^\mu u^\nu \leq 0$ for the timelike unit vector u^μ . Explicitly, in our ansatz with $(u_t, u_i) = (-\sqrt{f}, 0)$, it reads

$$T_t^t = \frac{1}{4e^2}N\phi'^2 + \frac{1}{4}m^2\Psi^2 + \frac{1}{4}fN\Psi'^2 + \frac{\mathcal{F}\phi^2}{4f} \geq 0. \quad (3.15)$$

Without the scalar field, the condition is fulfilled since $T_t^t = \frac{1}{4e^2}\phi'^2 \geq 0$. With the scalar field, the functions N, f, \mathcal{F} are positive outside the black hole, so the only term that can decrease the energy density T_t^t is due to the negative mass of the scalar field. If black hole hair is short, the scalar field decays fast as it goes to the boundary and is nontrivial close to the horizon. Let us analyze, therefore, the weak energy condition on the horizon. Using the gravitational equation (3.11), we obtain

$$T_t^t(r_+) = 24\pi T \frac{\sqrt{N(r_+)}}{r_+^3} - \frac{12}{\ell^4} \geq 0. \quad (3.16)$$

Our next step is to solve the above equations and determine the unknown functions $f(r)$, $N(r)$, $\phi(r)$ and $\Psi(r)$.

3.2 Boundary conditions for the fields

The equations of motion are not exactly solvable in a given ansatz when the scalar field is nonvanishing. To solve a system of second order differential equations, we have to specify

the behavior of the fields at the horizon, $r = r_+$, and at the asymptotic boundary, $r \rightarrow \infty$.

i) Behavior at the horizon ($r = r_+$)

We already discussed the behavior of the gravitational fields, which is

$$\begin{aligned} f(r_+) &= 0, & f'(r_+) &= \frac{4\pi T}{\sqrt{N(r_+)}} = \text{finite}, \\ N(r_+) &= \text{finite} \neq 0. \end{aligned} \quad (3.17)$$

For the electric potential, we choose the boundary condition

$$\phi(r_+) = 0, \quad (3.18)$$

which ensures a finite effective mass of the scalar field in the probe limit.

The scalar field has to be finite on the horizon. Using the equation of motion for the scalar field, we get

$$\Psi'(r_+) = \frac{m^2}{4\pi T} \frac{\Psi(r_+)}{\sqrt{N(r_+)}} = \text{finite}. \quad (3.19)$$

ii) Behavior at the boundary ($r \rightarrow \infty$)

The gravitational field (3.7) must be asymptotically locally AdS, so that we impose

$$\begin{aligned} f &\simeq k + \frac{r^2}{\ell^2} - \frac{M_0 \ell^2}{2r^2}, \\ fN &\simeq k + \frac{r^2}{\ell^2} - \frac{M \ell^2}{2r^2}, \\ N &= \frac{fN}{f} \simeq 1 + \frac{(M_0 - M) \ell^4}{2r^4}. \end{aligned} \quad (3.20)$$

The mass parameters are normalized so that, without the scalar field, they reduce to the known result of Sec. 2.1.2 with $M = M_0$. The black hole mass is related to the parameter M . The scalar hair, thus modifies the mass of the black hole through the function $N \neq 1$.

Related to the electric potential, the quantity of physical interest is the chemical potential $\mu = \phi(\infty) - \phi(r_+) = \phi(\infty)$ which represents the potential at infinity measured with respect to the event horizon. Based on the Maxwell equation, the electric potential behaves asymptotically as

$$\phi(r) \simeq \mu - \frac{\rho}{r^2}. \quad (3.21)$$

Finally, the scalar field must be finite everywhere so that it can be interpreted as the black hole hair. From the scalar equation (3.12), in the asymptotically AdS sector (3.20) and (3.21), we get

$$0 \simeq \Psi'' + \frac{5}{r} \Psi' - \frac{m^2 \ell^2}{r^2} \Psi, \quad (3.22)$$

leading to the asymptotic solution

$$\Psi(r) \simeq \frac{\Psi_-}{r^{\Delta_-}} + \frac{\Psi_+}{r^{\Delta_+}}. \quad (3.23)$$

Here, $\Delta_{\pm} = 2 \pm \sqrt{4 + m^2 \ell^2}$ is a conformal dimension of the scalar operators $\Psi_{\pm} = \langle \mathcal{O}_{\pm} \rangle$ in a dual conformal field theory (CFT).

A black hole with scalar hair forms when the background decreases the effective mass of the scalar field so that it becomes negative. This causes a breaking of the $U(1)$ gauge symmetry in the bulk gravity, which is dual to a condensation operator \mathcal{O} breaking the global $U(1)$ symmetry at the boundary [28].

A choice of the boundary conditions for the scalar field determines which quantity will be kept fixed on the boundary. If it is Ψ_- (Dirichlet boundary conditions), then the term Ψ_- becomes a source in the holographic quantum field theory (QFT) and a vacuum expectation value (VEV) of a scalar operator \mathcal{O}_+ of conformal dimension Δ_+ is identified with the bulk operator, $\Psi_+ = \langle \mathcal{O}_+ \rangle$. It is also possible to keep Ψ_+ fixed on the boundary (Neumann boundary conditions), when Ψ_+ becomes a source and $\Psi_- = \langle \mathcal{O}_- \rangle$ is the VEV.

On the other hand, the scalar field is finite when $\Delta_{\pm} \geq 0$, which implies that the dual operator is relevant or marginal and can be switched on without destroying the UV fixed point in CFT. The dual CFT is unitary for the masses of the scalar field that take values in the Breitenlohner-Freedman window, $-4 \leq m^2 \ell^2 \leq -3$, or equivalently $1 \leq \Delta_- \leq 2$, or $2 \leq \Delta_+ \leq 3$. The conformal anomaly is absent if $\Delta_{\pm} \neq 4$.

In the following discussion, we will set $m^2 \ell^2 = -3$, which corresponds to $\Delta_- = 1$ and $\Delta_+ = 3$, and choose the Neumann boundary condition where $\Psi_- = 0$. Then the response operator is $\Psi_+ = \langle \mathcal{O}_+ \rangle$ so that

$$\Psi(r) \simeq \frac{\langle \mathcal{O}_+ \rangle}{r^3}. \quad (3.24)$$

4 Black hole thermodynamics

Thermal properties of the black hole can be obtained from the partition function evaluated in semiclassical approximation,

$$Z = e^{-I_{\text{class}}^E}, \quad (4.1)$$

where I_{class}^E is the classical Euclidean action. In an asymptotically AdS spacetime this action is divergent and should be renormalized. Its finite part contains the thermodynamic information about the system through the quantum statistical relation

$$T I^E = U + \mu Q - T S, \quad (4.2)$$

where U is the total internal energy of the system at the temperature T , the constant Q is its electric charge and S is the black hole entropy. In the framework of AdS/CFT

correspondence, the Euclidean gravity action is identified with the thermodynamic potential (free energy) $G = TI^E$ of the holographic QFT. Also, the asymptotic charges in AdS space are interpreted as the thermodynamic charges in a boundary QFT. Finally, in the holographic dictionary, the black hole temperature and entropy match the field theory temperature and entropy, respectively.

Thus, in order to obtain the finite quantities in asymptotically AdS gravity (IR finiteness) and holographic QFT (UV finiteness), we need a renormalized gravitational action.

4.1 Renormalized action and Gibbs free energy

Euclidean spacetime (τ, r, y^m) is obtained from the Lorentzian spacetime (t, r, y^m) by performing the Wick rotation of the temporal coordinate ($t = i\tau$) with the period of the Euclidean time T^{-1} which avoids the conical singularity at the horizon. The Euclidean on-shell action $I^E = -iI$ has the form

$$I^E = \frac{V_3}{T} \int_{r_+}^{\infty} dr \frac{r^3}{\sqrt{N}} \mathcal{L}(r). \quad (4.3)$$

The volume $V_3 = \int d^3y \sqrt{\gamma} = \int \sinh^2 \eta \sin \theta d\eta d\theta d\phi$ of the hyperbolic transversal section is infinite, so all physical quantities are taken per unit volume.

Let us first evaluate the Euclidean bulk action, I_0 , which is a sum of the gravitational part (2.20) and the matter part (3.1). Substituting the equations of motion, all explicit contributions of the scalar field Ψ cancel out, so that the information about the black hole hair is contained in the function $N \neq 1$. Furthermore, the Euclidean on-shell action can be written as a total derivative,

$$\begin{aligned} I_0^E &= \frac{V_3}{2\kappa T} \int_{r_+}^{\infty} dr \left[12r\sqrt{N} f' (fN - k) + \frac{r^3}{e^2} \sqrt{N} \phi \phi' \right]' \\ &= \frac{V_3}{2\kappa T} \left[12r\sqrt{N} f' (fN - k) + \frac{r^3}{e^2} \sqrt{N} \phi \phi' \right] \Big|_{r_+}^{\infty}. \end{aligned} \quad (4.4)$$

Using the boundary conditions given in Sec. 3.2, one can see that I_0^E is divergent at infinity. We have not checked yet whether the action is stationary on-shell for the chosen boundary conditions.

Thus, we search for the suitable boundary terms for the gravitational (B_G) and matter (B_M) fields,

$$B = B_G + B_M, \quad (4.5)$$

so that the total action is convergent and that it has a well-posed action principle. We choose the Gauss-normal frame ($g_{ri} = 0$) in the local coordinates $x^\mu = (r, x^i)$ where

the boundary is placed at the constant radius, $r = r_B \rightarrow \infty$. Relevant quantities to describe the boundary dynamics are the induced metric, h_{ij} , and the extrinsic curvature, $K_{ij} = -h'_{ij}/2\sqrt{g_{rr}}$.

In a higher-order curvature AdS gravity, the simplest way to renormalize the gravitational action I_G is to add a unique boundary term, the so-called Kounterterm [42, 43], which depends explicitly on the extrinsic curvature. Its form in Einstein-Gauss-Bonnet AdS theory is given in Refs. [42, 44] and the PGB Kounterterm is obtained by leaving only the terms with the Gauss-Bonnet coupling,

$$B_G = \frac{3}{4\kappa} \int_{r_B} d^4x \sqrt{-h} \delta_{i_1 \dots i_4}^{j_1 \dots j_4} K_{j_1}^{i_1} \delta_{j_2}^{i_2} \left(\mathcal{R}_{j_3 j_4}^{i_3 i_4} - K_{j_3}^{i_3} K_{j_4}^{i_4} + \frac{1}{3\ell^2} \delta_{j_3}^{i_3} \delta_{j_4}^{i_4} \right), \quad (4.6)$$

where $\mathcal{R}_{jkl}^i(h)$ is the intrinsic curvature of the boundary. For more details on the method, see Refs. [43, 45, 46].

It is worthwhile noticing that the approach based on Kounterterms is equivalent to the standard one [47, 48] where the action principle based on Dirichlet boundary conditions for the boundary metric h_{ij} requires the PGB action to be supplemented by a generalized Gibbons-Hawking term,

$$B_{\text{GGH}} = \frac{2}{\kappa} \int_{r_B} d^4x \sqrt{-h} \delta_{i_1 i_2 i_3}^{j_1 j_2 j_3} K_{j_1}^{i_1} \left(\frac{1}{2} \mathcal{R}_{j_2 j_3}^{i_2 i_3} - \frac{1}{3} K_{j_2}^{i_2} K_{j_3}^{i_3} \right). \quad (4.7)$$

Indeed, the Dirichlet action $I_G + I_{\text{GGH}}$ has a variation proportional to δh_{ij} ,

$$\delta(I_G + B_{\text{GGH}}) = \frac{1}{\kappa} \int_{r_B} d^4x \sqrt{-h} \delta_{i_1 i_2 i_3}^{j_1 j_2 j_3} (h^{-1} \delta h)^i_j K_{j_1}^{i_1} \left(\frac{1}{2} \mathcal{R}_{j_2 j_3}^{i_2 i_3} - \frac{1}{3} K_{j_2}^{i_2} K_{j_3}^{i_3} \right), \quad (4.8)$$

because all variations δK_j^i cancel out. In consequence, $I_G + B_{\text{GGH}}$ vanishes for the Dirichlet boundary condition on the induced metric. However, this action is divergent and one has to add the counterterms that cancel the divergences and do not change the Dirichlet boundary condition (i.e., which depend only on the intrinsic quantities),

$$B_{\text{ct}} = \frac{1}{\ell\kappa} \int_{r_B} d^4x \sqrt{-h} \left(\frac{4}{\ell^2} - \mathcal{R} \right). \quad (4.9)$$

It can be shown in a near-boundary analysis that the surface terms $B_{\text{GGH}} + B_{\text{ct}}$ are equivalent to the Kounterterm B_G given by (4.6). We shall use the last one, as it is simpler and can be generalized to any dimension. (While the form of B_{GGH} is known in any d , the full series for B_{ct} is still unknown.) Evaluated on-shell, it becomes

$$B_G^E = \frac{9V_3}{2\kappa T} \lim_{r \rightarrow \infty} \sqrt{N} \left[f \left(\frac{r^2}{\ell^2} + 2k - fN \right) + \frac{rf'}{2} \left(2k + \frac{r^2}{3\ell^2} - 3fN \right) \right], \quad (4.10)$$

and in asymptotically AdS spaces it has the form

$$B_G^E = -\frac{6V_3}{\kappa T} \lim_{r \rightarrow \infty} \left(\frac{2r^4}{\ell^4} - \frac{3k^2}{4} - 2M + \frac{3M_0}{2} \right). \quad (4.11)$$

Then combining equations (4.4), (4.5) and (4.11), we have the renormalized action

$$I^E = \frac{V_3}{\kappa T} \left(\frac{9}{2} + 3M \right) + \frac{V_3 \mu \rho}{\kappa T e^2} - \frac{24\pi V_3}{\kappa} r_+ + B_M^E. \quad (4.12)$$

The matter fields do not contain IR divergences in the chosen coupling, but we can need the surface term B_M to ensure the stationary on-shell action.

Thus, let us check the variational principle. We already clarified that the gravitational action is based on Dirichlet boundary conditions for the induced metric. When we vary the matter field and use the equations of motion, we obtain

$$-\frac{1}{2\kappa} \int_{r_B} d^4x \sqrt{-h} n_r (F_{rt} \delta\phi - g^{rr} \Psi' \delta\Psi), \quad (4.13)$$

where $n_\mu = (n_r, n_i) = (\sqrt{f}, 0)$ is the unit normal to the asymptotic boundary. The above expression must vanish or be canceled out by the boundary term B_M . For the electromagnetic field, we can choose a grand canonical ensemble where the chemical potential is kept fixed on the boundary, $\delta\phi = 0$, which will make the first term in (4.13) vanish. Then well-defined variation principle for the Maxwell field does not require any surface term. Alternatively, if one considers a canonical ensemble where the charge density is kept fixed, $\delta\phi' = 0$, a new term, $\frac{1}{2\kappa e^2} \int_{r_B} d^4x \sqrt{-h} n_r \phi F^{tr}$ has to be added.

Similarly, there are at least two possible choices of a boundary condition for the scalar field: the Dirichlet condition ($\delta\Psi = 0$) and the Neumann one ($\delta\Psi' = 0$). In the former case, one does not need new boundary terms in the action to have its variation well defined, while in the latter case one should add $-\frac{1}{2\kappa} \int_{r_B} d^4x \sqrt{-h} n^r \Psi \Psi'$. Holographically, these two choices correspond to different quantizations in the quantum field theory, with $\Psi_- = 0$ or $\Psi_+ = 0$ in Eq. (3.23), identifying the scalar field with the VEV or the source, respectively.

In this work, the mass of the scalar field is set to $m^2 \ell^2 = -3$. We shall also choose the Neumann boundary conditions for the scalar field, making the source vanish, $\Psi_- = 0$. With respect to the electromagnetic field, we shall work in a grand canonical ensemble by fixing the chemical potential. Thus, the boundary term for the matter fields reads

$$B_M = -\frac{1}{2\kappa} \int_{r_B} d^4x \sqrt{-h} n^r \Psi \Psi'. \quad (4.14)$$

Evaluating it on-shell we find that, due to a fast falloff of $\Psi(r)$, the scalar field boundary term does not contribute, $B_M^E = 0$.

In order to interpret the expression for the free energy (4.12), we have to calculate the conserved charges and entropy of the system.

4.2 Conserved charges

Total internal energy of the black hole can be obtained as the Noether charge associated with the asymptotic Killing vector $\xi^i = \delta_t^i$ for time translations, evaluated at the transversal section of spacetime, $t, r = \text{Const}$, denoted by Σ_r , through the formula [42]

$$U = \int_{\Sigma_\infty} d^3y \sqrt{\sigma} u_j \xi^i \left(q_{(0)i}^j + q_i^j \right). \quad (4.15)$$

Here, $u_i = -\sqrt{-g_{tt}} \delta_i^t$ is the unit normal to the surface Σ_r described by the transversal metric $\sigma_{mn} = r^2 \gamma_{mn}$ and γ_{mn} is the hyperbolic metric given by Eq.(3.8). It is well known that in five dimensions the total gravitational energy includes the vacuum energy ($q_{(0)i}^j$ term), black hole mass M_{BH} (gravitational q_i^j contribution) and energy of the matter fields (matter q_i^j contribution), that is,

$$U = E_{\text{vac}} + M_{\text{BH}} + M_{\text{M}}. \quad (4.16)$$

The vacuum energy exists in odd bulk dimensions only, and it corresponds to the energy of the empty (global) AdS space. In particular, the vacuum energy in Einstein-Gauss-Bonnet (EGB) gravity with topological black holes is given in Ref. [44], from where we can deduce the five-dimensional PGB expression by keeping only the Gauss-Bonnet contribution,

$$E_{\text{vac}} = \int_{\Sigma_\infty} d^3y \sqrt{\sigma} u_t q_{(0)t}^t = \frac{V_3}{\kappa} \frac{9k^2}{2}. \quad (4.17)$$

On the other hand, it was shown in Refs. [49, 50] that the gravitational part of the charge density tensor q_i^j in EGB gravity can be written in terms of the Weyl tensor, $W_{\alpha\beta}^{\mu\nu} = R_{\alpha\beta}^{\mu\nu} - \frac{4}{3} \delta_{(\alpha}^{(\mu} R_{\beta)}^{\nu)} + \frac{1}{12} \delta_{\alpha\beta}^{\mu\nu} R$ or, more precisely, its electric part $W_{j\nu}^{i\mu} n_\mu n^\nu = W_{jr}^{ir}$ as

$$M_{\text{BH}} = \frac{1}{2\kappa\ell} \int_{\Sigma_\infty} d^3y \sqrt{\sigma} u^j \xi^i W_{jr}^{ir}, \quad (4.18)$$

where $n_\mu = \delta_\mu^r (fN)^{-1/2}$ is the normal vector to the spacetime boundary $r = \text{Const}$. In fact, since the Weyl tensor is traceless, the quantity that enters the black hole mass is

$$W_{jr}^{ir} = -W_{jk}^{ik} = - \left(R_{jk}^{ik} + \frac{1}{\ell^2} \delta_{jk}^{ik} \right) + \mathcal{O}(1/r^8), \quad (4.19)$$

so that it can be easily evaluated with the help of (3.20) as

$$M_{\text{BH}} = \frac{6}{\kappa \ell} \int_{\Sigma_\infty} d^3y r^3 \sqrt{\gamma} \sqrt{f} \left(\frac{fN - k}{r^2} - \frac{1}{\ell^2} \right) = \frac{V_3}{\kappa} 3M. \quad (4.20)$$

We can also show that $M_{\text{M}} = 0$, so the total internal energy of the black hole (the vacuum energy plus its mass) is

$$U = \frac{V_3}{\kappa} \left(\frac{9k^2}{2} + 3M \right). \quad (4.21)$$

The action is also invariant under the local $U(1)$ transformations $\delta_\lambda A_\mu = \partial_\mu \lambda$, $\delta_\lambda \theta = \lambda$ (ψ , $g_{\mu\nu}$ do not transform). The Noether charge is calculated from the electromagnetic current $J^\mu(\lambda) = \frac{1}{2\kappa e^2} \partial_\nu (\lambda \sqrt{-g} F^{\mu\nu})$ with $\lambda = 1$,

$$Q = \int_{r_B} d^4x J^r = \frac{V_3}{2\kappa} \lim_{r \rightarrow \infty} \left(r^3 \sqrt{N} \phi' \right) = \frac{V_3 \rho}{\kappa e^2}. \quad (4.22)$$

Found Noether charges U and Q should match the thermodynamic charges

$$\begin{aligned} U &= G - T \left(\frac{\partial G}{\partial T} \right)_\mu - \mu \left(\frac{\partial G}{\partial \mu} \right)_T, \\ Q &= \left(\frac{\partial G}{\partial \mu} \right)_T, \end{aligned} \quad (4.23)$$

obtained from the thermodynamic partition function using the first law of thermodynamics. The thermodynamic charges are the ones that enter the quantum statistical relation.

4.3 Quantum statistical relation

To describe a thermodynamic system, we have to know its entropy. The entropy of Lovelock AdS gravities is given, for example, in Ref. [51]. Applying this formula for static, spherically symmetric, topological black holes in pure Lovelock gravity with the coupling constant α_p , we obtain

$$S = \frac{(d-1)! V_{d-1} p \alpha_p}{4G (d-2p+1)!} r_+^{d-2p+1} k^{p-1}. \quad (4.24)$$

In the PGB case ($p = 2$) in five dimensions ($d = 4$) with $16\pi G = 2\kappa$ and $\alpha_2 = -1$, and with the hyperbolic horizon ($k = -1$), the above expression becomes

$$S = \frac{24\pi V_3}{\kappa} r_+. \quad (4.25)$$

It is worthwhile noticing that S is positive only for the hyperbolic horizons, which are the only black holes that exist in the PGB gravity in AdS space.

Now we can interpret Eq.(4.12) in the grand canonical ensemble ($B_M^E = 0$) as the quantum statistical relation of the system (4.2). Indeed, replacing the expressions for the total energy (4.21), the electric charge (4.22) and the entropy (4.25), we obtain

$$G = TI^E = U + \mu Q - TS, \quad (4.26)$$

which is nothing but the Legendre transformation of the Gibbs potential G . Then the first law of thermodynamics, $\delta U = T\delta S - \mu\delta Q$, can be equivalently written as $\delta G = Q\delta\mu - S\delta T$.

Let us discuss the case $\Psi = 0$ where the exact solution is known. From $\phi(r_+) = 0$ and $f(r_+) = 0$, we calculate the charges,

$$\begin{aligned} M_0 &= \frac{r_+^4}{\ell^4} - 1 - \frac{\mu^2 r_+^2}{6e^2}, \\ Q &= \frac{V_3 \mu r_+^2}{\kappa e^2}, \end{aligned} \quad (4.27)$$

and Eq.(2.23) gives the temperature

$$T = \frac{1}{4\pi} \left(\frac{2r_+^3}{\ell^4} + \frac{\mu^2 r_+}{6e^2} \right). \quad (4.28)$$

Using the expression (4.21) for the internal energy, we obtain the free energy in terms of r_+ and μ [because its natural variables are $T(r_+, \mu)$ and μ],

$$G = \frac{V_3}{\kappa} \left(\frac{3}{2} - \frac{9r_+^4}{\ell^4} - \frac{\mu^2 r_+^2}{2e^2} \right). \quad (4.29)$$

It is straightforward to show by varying G in r_+ and μ that the first law of thermodynamics is fulfilled.

In the next section we perform a similar calculation for the case $\Psi \neq 0$.

5 Holographic phase transition

Having an asymptotically AdS space and the scalar field turned on, we can use the AdS/CFT correspondence tools to study a dual quantum theory. A particular black hole solution breaks the conformal symmetry on the boundary and leads to a holographic theory which is thermal. Our goal is to analyze the possibility of having a phase transition in the four-dimensional QFT due to a change of temperature. In practice, this means that we have to find a backreaction solution of the system. Since it is not exactly solvable,

we shall integrate numerically a set of equations (3.10)-(3.13) and use the probe limit to simplify it. Namely, in this limit the gravity dynamically decouples from the matter, and the scalar field moves in the black hole background.

In addition, when the mass of the scalar field saturates the upper Breitenlohner-Freedman bound, the gravitational backreaction could modify the asymptotic behavior of the theory, and the free energy in asymptotically AdS space would require additional surface terms in order to become regular [52]. Thus, the dynamics without the gravitational backreaction better catches a typical behavior of the system.

For numerical calculations it is convenient to introduce a new dimensionless variable, $z = r_+/r$. All functions are defined in the region $z \in (0, 1]$, where $z = 1$ is the location of the horizon and $z = 0$ corresponds to the asymptotic boundary. We can set $r_+ = 1$ by the following rescaling,

$$\begin{aligned} r &\rightarrow r_+ r, & \Psi &\rightarrow r_+ \Psi, & \ell &\rightarrow \ell/r_+, \\ f &\rightarrow r_+^2 f, & \phi &\rightarrow r_+^2 \phi, & m &\rightarrow r_+ m, \\ N &\rightarrow r_+^2 N, & \mathcal{F} &\rightarrow r_+^2 \mathcal{F}, & k &\rightarrow r_+^4 k. \end{aligned} \quad (5.1)$$

The probe approximation is obtained as the large charge limit, $e \rightarrow \infty$, after rescaling $\Psi = \frac{1}{e} \tilde{\Psi}$ in the Eqs. (3.10)-(3.13) [41]. The scaling properties of the function $\mathcal{F}(\Psi)$ are determined from the behavior of Ψ^n , whose dimensional analysis gives $\mathcal{F} \sim c_n (eL)^{n-2} \Psi^n$, where L is some length scale and c_n a dimensionless constant, implying that $\mathcal{F}(\frac{1}{e} \tilde{\Psi}) \rightarrow 0$ when $e \rightarrow \infty$. In general, we shall require that

$$\begin{aligned} \tilde{\mathcal{F}}(\tilde{\Psi}) &= \lim_{e \rightarrow \infty} \left[e^2 \mathcal{F} \left(\frac{1}{e} \tilde{\Psi} \right) \right] < \infty, \\ \tilde{\mathcal{F}}'(\tilde{\Psi}) &= \lim_{e \rightarrow \infty} \left[e^2 \frac{d}{d\tilde{\Psi}} \mathcal{F} \left(\frac{1}{e} \tilde{\Psi} \right) \right] < \infty. \end{aligned} \quad (5.2)$$

Then the gravitational equations (3.10) and (3.11) solve $N = 1$ and (3.11)-(3.13) in the probe limit become

$$\begin{aligned} 0 &= \frac{6f'(f-k)}{r^3} - \Lambda, \\ 0 &= \frac{(r^3 f \Psi')'}{r^3} - m^2 \Psi + \frac{\phi^2 \mathcal{F}'}{2f}, \\ 0 &= \frac{(r^3 \phi')'}{r^3} - \frac{\mathcal{F} \phi}{f}, \end{aligned} \quad (5.3)$$

where we drop tildes for the sake of simplicity. The gravitational PGB solution is a neutral hyperbolic black hole with the metric function

$$f(r) = -1 + \frac{r^2}{\ell^2} \sqrt{1 - \frac{M_0 \ell^4}{r^4}}, \quad (5.4)$$

which is just Eq. (2.21) with the electric charge switched off. The boundary conditions are the same as the ones presented in Sec. 3.2, where now $N = 1$ and thus $M = M_0$.

Since the gravity part has been decoupled, we have to focus only on the matter action which, in the black hole background, has the form

$$I_M = \frac{1}{4\kappa e^2} \int d^5x \sqrt{\gamma} r^3 \left(-\phi'^2 + f\Psi'^2 + m^2\Psi^2 - \frac{1}{f} \phi^2 \mathcal{F} \right). \quad (5.5)$$

In the grand canonical ensemble and with the scalar field satisfying the Neumann boundary conditions, we have to add the matter boundary term (4.14). Using the equations of motion, the Euclidean continuation of $I_M + B_M$ is

$$I_0^E + B_M^E = \frac{V_3}{4\kappa e^2 T} \left[\left(r^3 f \Psi \Psi' + r^3 \phi \phi' \right) \Big|_{r_+}^{\infty} - \int_{r_+}^{\infty} dr \frac{r^3 \phi^2 \Psi \mathcal{F}'}{2f} \right]. \quad (5.6)$$

Compared to the system that includes the backreaction, the on-shell action is not a total derivative, and the nonlocal term has to be evaluated numerically between the horizon and the asymptotic boundary. Another important difference with respect to the backreaction noted in Ref. [35], is that the action needs scalar field counterterms when evaluated in the probe limit, since it becomes IR divergent. This counterterm is discussed in Ref. [41] and it does not contribute to the result for our boundary conditions, as we are allowed to set $\Psi_- = 0$.

The finite free energy, $G = T(I_0^E + B_M^E)$, has the form

$$G = \frac{V_3}{2\kappa e^2} \left(\rho\mu - \int_{r_+}^{\infty} dr \frac{r^3 \phi^2 \Psi \mathcal{F}'}{4f} \right). \quad (5.7)$$

We use the shooting method to solve the equations of motion

$$\begin{aligned} 0 &= \Psi'' + \left(\frac{f'}{f} - \frac{1}{z} \right) \Psi' + \frac{3\Psi}{z^4 f} + \frac{\phi^2 \mathcal{F}'(\Psi)}{2z^4 f^2}, \\ 0 &= \phi'' - \frac{1}{z} \phi' - \frac{\mathcal{F}\phi}{z^4 f}, \end{aligned} \quad (5.8)$$

from the horizon to the boundary. Numerical results show that the black hole will develop scalar hair below some critical temperature T_c , because a nontrivial solution for the scalar field appears. A change of strength of the condensation with the temperature is shown in Figure 2. The critical temperature is around $T_c \simeq 0.05615\mu$. Its value is not sensitive to the coupling parameters choice. A difference between the free energies of the condensation state and the normal state is shown in Figure 3. The free energy is lower than the one in the normal state ($G_{\text{superconducting}} \leq G_{\text{normal}}$), which implies that the phase transition will occur. Furthermore, without the cubic interaction ($c_3 = 0$), the second order phase

transition turns to the first order at $c_4 \sim 0.2$, while without the quartic term ($c_4 = 0$), the phase transition becomes first order if $c_3 > 0$. This critical value of c_4 is lower than the ones in Einstein gravity, while a positive c_3 leads to the first order phase transition, which agrees with that found in Einstein gravity [53]. In addition, the lines of the first order phase transition in the right plot of Figure 3 are similar to the ones shown in Einstein gravity. However, the dashed line in the left plot is a bit different because both possible hairy solutions have lower free energies than the normal phase. This does not affect the final conclusion because only the solution with the lowest free energy will be physically realized.

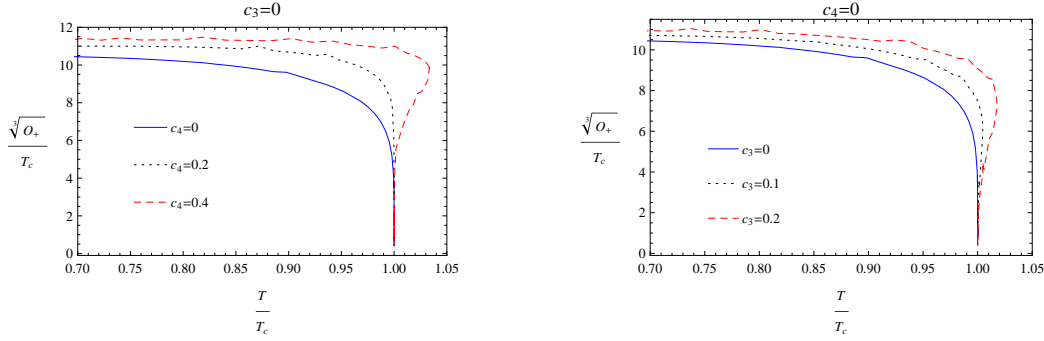


Figure 2: Values of the condensate $\langle \mathcal{O}_+ \rangle$ in the probe limit.

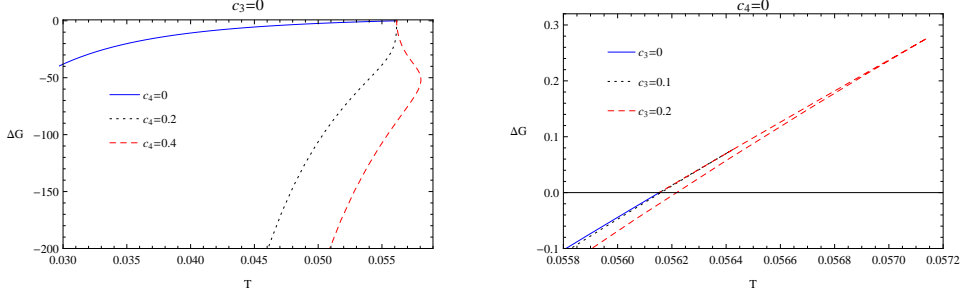


Figure 3: A difference between the free energies of the hairy phase and the normal phase in the probe limit.

We conclude that, as long as the relative strengths of the electromagnetic and gravitational couplings are such that the backreaction of the gravitational field can be neglected, the QFT dual to the PGB AdS gravity behaves similarly as the QFT dual to the Einstein AdS gravity, and it exhibits a phase transition of the first or second order. It is worth-

while to explore this phenomenon in a larger range of interactions, so that it includes a backreaction. We hope to solve this problem elsewhere in the future.

6 Conclusions

We generalized a static, spherically symmetric solution for neutral black holes in pure Lovelock gravity discussed in Refs. [14, 15] into electrically charged, topological black holes, which have the same falloff as the RN-AdS solution far from the matter source. In the particular case of PGB gravity, we analyzed its thermodynamical behavior in the grand canonical ensemble, based on the renormalized Euclidean action and the quantum statistical relation. We used the Noether charges for this purpose, but we showed that they coincide with the thermodynamic charges. The entropy for this system with the hyperbolic horizon grows linearly with the increase of the radius, and the temperature grows as r_+^3 for large black holes, or $1/r_+^3$ for the small ones. The extremal black holes do not exist and the horizon forms for any value of the mass, including zero, as long as there is a nontrivial electric charge. In spite of this unusual behavior, typical for pure Lovelock gravity, we showed that the first law of thermodynamics is satisfied.

We produced the quantum statistical relation from the renormalization of the on-shell bulk action in the presence of the scalar field satisfying Neumann boundary conditions, where we found that the scalar field did not enter explicitly the expression for the thermodynamic potential. Influence of the scalar field is contained in the values of the black hole parameters.

We also explored the possibility of having a hyperbolic holographic superconductor with a Stückelberg correction dual to charged pure Gauss-Bonnet gravity. We found that, as in the Einstein-Gauss-Bonnet case, there is a hairy black hole solution below some critical temperature T_c . This temperature is lower than in PGB gravity, compared to similar settings in the Einstein-Hilbert or Einstein-Gauss-Bonnet cases, due to a higher-order kinetic term. With the increase of the Stückelberg parameters c_3 and c_4 , the hairy solution becomes stronger while the critical temperature is not affected. A numerical analysis showed that the hairy state of the PGB AdS black hole has lower energy than the black hole without the scalar field, only if the electric coupling is large enough that the backreaction of the gravitational field can be neglected. In that case, the phase transition occurs in a dual field theory below the critical point.

ACKNOWLEDGMENTS

The authors would like to thank Gaston Giribet, Rodrigo Olea and Ricardo Troncoso for their valuable comments. L.A. is grateful to Naresh Dadhich for the hospitality and the useful discussions during her stay at Inter-University Centre for Astronomy and Astrophysics (IUCAA). This work was funded by FONDECYT Grant No.3150006 and PUCV-DI Projects No. 123.736/2015 and No. 123.738/2015. L.A. is financed in part by the MECESUP project FSM1204 2-2014 and the UTFSM Grant No. PIIC 2015. X.-M. Kuang is also partly supported by the ARISTEIA II action of the operational programme education and long life learning which is co-funded by the European Union (European Social Fund) and National Resources.

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